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A Note on Diophantine Equation $Y^2 + k = X^5$

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Abstract. We show that if the class number of the quadratic field $A(\sqrt{-k})$ is not divisible by 5, and if k is not congruent to 7 modulo 8, then the equation $Y^2 + k = X^5$ has no solutions in rational integers X, Y with the exception of $k = 1, 19, 341$.

In this paper we shall discuss the solutions in integers of the diophantine equation

$$Y^2 + k = X^5, \tag{1}$$

where k is a positive square-free integer. We shall show that if the class number of the quadratic field $Q(\sqrt{-k})$ is not divisible by 5, and if k is not congruent to 7 modulo 8, then Eq. (1) has no solutions with three exceptions. We will also find all the solutions of Eq. (1) in the exceptional cases. Results of this note are announced in [6].

1. Application of the Arithmetic of the Ideals in $Q(\sqrt{-k})$.

Throughout this section we shall assume that X and Y are integers which satisfy Eq. (1). Passing to the ideals in the quadratic field $Q(\sqrt{-k})$, we find that

$$[Y + \sqrt{-k}] = C_1 A^5, \quad [Y - \sqrt{-k}] = C_2 B^5, \tag{2}$$

where C_1 and C_2 are free of fifth powers; and that

$$[X]^5 = C_1 C_2 A^5 B^5. \tag{3}$$

Now, any prime divisor P of C_1 must also divide C_2; and any common prime divisor of C_1 and C_2 must divide $2\sqrt{-k}$. Let $V_P(\alpha)$ denote the highest power of the prime ideal P dividing α.

LEMMA. *If the prime ideal P in the quadratic field $Q(\sqrt{-k})$ is invariant under the conjugation automorphism σ, where $\sigma(a + b\sqrt{-k}) = a - b\sqrt{-k}$, then P does not divide C_1.*

Proof. Now $\sigma(C_1) = C_2$, so that $V_P(C_1) = V_{\sigma(P)}(C_2) = V_P(C_2)$, since $\sigma(P) = P$. In addition, $V_P(2\sqrt{-k}) \leqslant 3$ for any prime ideal P. Furthermore, $C_1 C_2$ is a fifth power, by (3). Hence, $V_P(C_1 C_2) = 2V_P(C_1) = 0, 2, 4$ or 6 and $V_P(C_1 C_2) \equiv 0 \pmod 5$. Thus, $V_P(C_1) = 0$.

COROLLARY. *If* $k \not\equiv 7 \pmod 8$, *then* $[y + \sqrt{-k}]$ *is a fifth power.*

Proof. Any odd prime P dividing C_1 divides $\sqrt{-k}$; hence P ramifies, and so we have $P = [p]^2$ where p is a prime number. For such P, it is clear that $\sigma(P) = P$. If P is a prime dividing 2, then P ramifies if $k \not\equiv 3 \pmod 8$ and 2 remains prime if

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$k \equiv 3 \pmod 8$; we then have either $\mathfrak{P} = [p]^2$ or $\mathfrak{P} = [2]$, both of which are invariant under σ. Hence, if $k \not\equiv 7 \pmod 8$, then $[y + \sqrt{-k}]$ is a fifth power.

As a consequence, we have the following:

THEOREM 1. *If the positive square-free integer k satisfies* $k \not\equiv 7 \pmod 8$ *and if* $h(-k)$, *the class number of* $Q(\sqrt{-k})$, *is prime to* 5, *then the equation* $y^2 + k = X^5$ *is equivalent to*

$$y + \sqrt{-k} = \alpha^5, \tag{4}$$

where α *is an integer in* $Q(\sqrt{-k})$.

Proof. By the corollary, $[y + \sqrt{-k}] = \mathfrak{A}^5$; since $(h(-k), 5) = 1$, if the fifth power of an ideal is principal, then the ideal itself is principal. Thus, $y + \sqrt{-k} = \epsilon\alpha_1^5$ where α_1 is an integer of $Q(\sqrt{-k})$ and ϵ is a unit of $Q(\sqrt{-k})$. Since every unit of every imaginary quadratic field is a fifth power, we must have $y + \sqrt{-k} = \alpha^5$, where α is an integer of $Q(\sqrt{-k})$.

2. **The Equation** $y + \sqrt{-k} = \alpha^5$; $k \not\equiv 3 \pmod 4$. By the above, we put $\alpha = A + B\sqrt{-k}$ where A and B are rational integers. Then $X = \text{Norm}(\alpha) = A^2 + kB^2$, and if we equate imaginary parts of $y + \sqrt{-k} = \alpha^5$, we find that

$$5A^4B - 10A^2B^3k + B^5k^2 = 1. \tag{5}$$

Hence, $B = \pm 1$ and we have

$$k^2 - 10A^2k + (5A^4 \pm 1) = 0. \tag{6}$$

Thus,

$$k = \tfrac{1}{2}(10A^2 \pm \sqrt{100A^4 - 4(5A^4 \pm 1)}) = \tfrac{1}{2}(10A^2 + \sqrt{\Delta}).$$

Hence, $\Delta = 4(20A^4 \pm 1)$ must be a square, i.e.

$$20A^4 \pm 1 = l^2. \tag{7}$$

Now $20A^4 - 1 = l^2$ has no solutions modulo 4; the equation $20A^4 + 1 = l^2$ has only the solutions $(l, A) = (1, 0)$ and $(161, \pm 6)$ (see [3], [4] and [5]). These yield the values $k = 1, 341$ and 19 ($k = 19$ will be used in the next section); their corresponding solutions to (1) are

$$0^2 + 1 = 1^5 \quad \text{and} \quad (2759646)^2 + 341 = (377)^5.$$

3. **The Equation** $Y + \sqrt{-k} = \alpha^5$; $k \equiv 3 \pmod 8$. We may assume that $\alpha = \tfrac{1}{2}(A + B\sqrt{-k})$ where $A \equiv B \pmod 2$ are rational integers. Comparing the imaginary parts as before, we have

$$5A^4B - 10A^2B^3k + B^5k^2 = 32. \tag{8}$$

If $B = \pm 1$, then

$$k^2 - 10A^2k + 5A^4 = \pm 32, \tag{9}$$

so that

$$k^2 \equiv \pm 2 \pmod 5, \tag{10}$$

which is impossible. Thus, $B = 2B_1$; this implies, due to parity considerations, that $A = 2A_1$; and we are led to the equation

$$5A_1^4B_1 - 10A_1^2B_1^3k + B_1^5k^2 = \pm 1. \tag{11}$$

Hence, $B_1 = \pm 1$ and we are led to the equation

$$k^2 - 10A_1^2k + (5A_1^4 \pm 1) = 0, \tag{12}$$

which was treated in Section 2. The only solution satisfying $k \equiv 3 \pmod 8$ was found to be $k = 19$, which yields

$$(22434)^2 + 19 = (55)^5.$$

In summary, we have:

THEOREM 2. *If k is a square-free positive integer for which $k \not\equiv 7 \pmod 8$ and $(h(-k), 5) = 1$, then the equation $Y^2 + k = X^5$ has only the solutions* $(k, Y, X) = (1, 0, 1), (19, \pm 22434, 55)$ *and* $(341, \pm 275964, 377)$.

Remark. There are 49 square-free integers less than 100 and incongruent to 7 (mod 8); of these, only $k = 74$ and 86 do no satisfy $(h(-k), 5) = 1$.

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